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A converse to the Eidelheit theorem in real Hilbert spaces

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Abstract

We establish the following converse to the Eidelheit theorem: an unbounded closed and convex set of a real Hilbert space may be separated by a closed hyperplane from every other disjoint closed and convex set, if and only if it has a finite codimension and a non-empty interior with respect to its affine hull.

Keywords: Convex separation; Hahn–Banach theorem; Eidelheit theorem; Compactly epi-lipschitzian sets

1. Introduction and notation

The Hahn–Banach theorem (the crown jewel of functional analysis, as called in [9]) is undoubtedly one of the most elegant and powerful results in functional analysis. Among its most important consequences are the convex separation theorems, also called the geometric forms of the Hahn–Banach theorem. These results say that under various assumptions, it

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is possible to separate by a closed hyperplane a pair C_1 and C_2 of convex subsets of a topological vector space X with topological dual X^*. In other terms,

$$\exists f \in X^* \text{ s.t.} \quad \sup_{x \in C_1} \langle f, x \rangle \leqslant \inf_{x \in C_2} \langle f, x \rangle.$$

In the real Banach reflexive setting, since each bounded closed and convex set is compact for the weak topology, any two disjoint closed and convex sets can be separated by a closed hyperplane provided that one of the sets is bounded. In this context, let us also recall the result of James (see [8]) which states that a real Banach space is reflexive if and only if each pair (C_1, C_2) of disjoint closed convex subsets, one of which is bounded, can be strictly separated by a closed hyperplane, i.e.,

$$\exists f \in X^*, \exists \alpha \in \mathbb{R} \text{ s.t.} \quad \sup_{x \in C_1} \langle f, x \rangle < \alpha < \inf_{x \in C_2} \langle f, x \rangle.$$

(For further developments of this topic, the reader is referred to [6].)

However, this convex separation result does no longer hold when the closed convex sets to be separated are both unbounded. Exploiting the notion of an *internal point* of a convex set C (that is a point $x \in C$ such that for every $v \in X$ there exists $\varepsilon > 0$ such that $x + tv \in C$ for every $-\varepsilon \leqslant t \leqslant \varepsilon$), the Eidelheit theorem (see [5]) states in the setting of locally convex spaces, that a pair of closed and convex sets may be strictly separated by a closed hyperplane, provided one of them has a non-empty interior. A slightly more general version of this theorem was proved in the setting of normed spaces in [4, Theorem 3.2], by using the notion of a compactly epi-lipschitzian (CEL) set C: for every $x \in C$ there exist reals $r, s, \varepsilon > 0$ and a compact subset K of X such that

$$\big(C \cap (x + r\mathbb{B}_X)\big) + \lambda s \mathbb{B}_X \subseteq C + \lambda K, \quad \forall \lambda \leqslant \varepsilon.$$

The authors proved that the Eidelheit separation property still holds for closed and convex CEL sets as a consequence of their main result [4, Theorem 2.5], which states that a closed and convex set C is CEL is and only if its relative norm-interior, that is the norm-interior relative to the closed affine hull of C, is non-empty, while its codimension, that is the codimension of the affine hull of C, is finite.

Let us remark that, when the underlying space X is finite dimensional, the above mentioned version of the Eidelheit theorem implies that it is possible to separate every two disjoint closed and convex sets (see for instance Theorem 11.3 in [10]). For a more detailed account, the reader is referred to the monograph [8] (see also [11]). In [9] is given a survey of the recent developments in Hahn–Banach theory.

In this paper we shall be concerned with the converse to the Eidelheit theorem as stated in [4, Theorem 3.2] (in the framework of real Hilbert spaces). Our main result (Theorem 2) states that an unbounded closed and convex set of a real Hilbert space X may be separated by a closed hyperplane from every other disjoint, closed and convex set if and only if it is compactly epi-lipschitzian, that is (in harmony with [4, Theorem 2.5]) if and only if it has a finite codimension and a non-empty relative norm-interior. In other words, we prove, in this context, that the two conditions from Eidelheit theorem are not only sufficient, but also necessary.

Thus, given an unbounded non-empty closed convex set C such that either its codimension is finite or its relative norm interior is empty, the main goal of this paper is to construct a closed and convex set D disjoint from C such that

$$\sup_{x\in D}\langle f,x\rangle > \inf_{x\in C}\langle f,x\rangle \quad \forall f\in X,\ f\neq 0.$$

We first consider two particular cases: in Section 3, C is an *unbounded non-empty closed convex set with finite dimension*, and in Section 4, C is an *unbounded closed set with empty norm interior which spans a separable Hilbert space*. Finally, by using some analytical tools developed in Section 2, a dimension reduction result (Lemma 4, the analysis provided in Section 5) makes it possible to reduce the general case to one of the two cases studied in Sections 3 and 4.

Throughout the paper, we suppose that X is a real Hilbert space with closed unit ball $\mathbb{B}_X$. The norm of X space denoted by $\|\cdot\|$ is associated to a scalar product $\langle\cdot,\cdot\rangle$.

As usual, given a subset S of X,

$$S^p = \{f\in X\colon \langle f,w\rangle \leqslant 1, \forall w\in S\}$$

will denote the *polar set* of S and we note

$$S^\circ = \{f\in X\colon \langle f,w\rangle \leqslant 0, \forall w\in S\},$$

the *negative polar cone* of S (remark that $S^p = S^\circ$ when S is a cone), and notice that S° reduces to the *orthogonal* complement

$$S^\perp = \{f\in X\colon \langle f,w\rangle = 0, \forall w\in S\}$$

when S is a linear subspace of X. Set also $S_r = S\cap r\mathbb{B}_X$ for every $r>0$.

We recall that the *recession cone* (see [10]) to the closed convex set S is the closed convex cone S^∞ defined by

$$S^\infty = \{v\in X\colon \forall\lambda>0,\ \forall x_0\in S,\ x_0+\lambda v\in S\},$$

and that a set S is called *linearly bounded* whenever $S^\infty=\{0\}$. The linear subspace of X parallel to the largest linear manifold contained in S will be denoted by $l(S)$:

$$l(S) = S^\infty \cap (-S^\infty).$$

Let us also note by $\overline{\mathrm{sp}}(S)$ the closed linear span of the set S, i.e., $\overline{\mathrm{sp}}(S)$ is the smallest closed linear space containing S, and recall that S is said to span X if $\overline{\mathrm{sp}}(S)=X$.

Given a closed convex subset S of X, the *barrier cone* of S is defined as follows:

$$\mathcal{B}(S) = \{f\in X\colon \sup_{x\in S}\langle f,x\rangle < +\infty\}.$$

Finally, we use the symbols "$\to$" and "$\rightharpoonup$" to denote the strong convergence and the weak convergence on X, and Int S for the topological norm interior of a given set S.

2. Technical preliminaries

Separation results which will be obtained in the following sections heavily rely on several technical results which will be presented in this section. The first one is a classical argument, valid only in separable settings (see [7]; a detailed review of the question, in connection with the notion of quasi-relative interior can be found in Borwein and Lewis [3]).

Theorem 1. *Let X be a separable real Hilbert space and C be a closed convex subset of X. Then C spans X if and only if there exists a point $\bar{a} \in C$ such that the relation*

$$\inf_{x\in C}\langle f, x\rangle < \langle f, \bar{a}\rangle < \sup_{x\in C}\langle f, x\rangle$$

holds for every $f \in X$, $f \neq 0$.

The following proposition provides a necessary and sufficient condition for a non-empty closed convex set spanning an Hilbert space to have a non-empty norm-interior.

Proposition 1. *Let C be a closed and convex set which spans an infinite dimensional real Hilbert space X. The two following statements are equivalent*:

(a) *the norm-interior of C is empty*;

(b) *there exists a sequence $(f_n)_{n\in\mathbb{N}^*} \subset X$, $\|f_n\| = 1$, such that*

$$f_n \rightharpoonup 0 \quad \textit{and} \quad \lim_{n\to\infty}\Bigl(\sup_{x\in C}\langle f_n, x\rangle\Bigr) = 0. \tag{1}$$

Proof. Consider

$$C^p = \bigl\{f \in X\colon \langle f, x\rangle \leqslant 1, \forall x \in C\bigr\},$$

the polar set of C. Let us remark that

$$\bigl[f \in l(C^p)\bigr] \quad \Leftrightarrow \quad \bigl[\langle f, x\rangle = 0, \forall x \in C\bigr];$$

as C spans X, it follows from Theorem 1 that $l(C^p) = \{0\}$.

Let us estimate the barrier cone $\mathcal{B}(C^p)$ of the polar set C^p. As for every set K, $\mathcal{B}(K) = \bigcup_{r>0} rK^p$, it follows that $\mathcal{B}(C^p) = \bigcup_{r>0} r(C^p)^p$. It is well known (see for instance [10, p. 125]) that $(C^p)^p = \bigcup_{0\leqslant s\leqslant 1} sC$, and thus

$$\mathcal{B}(C^p) = \bigcup_{r>0} r\Biggl(\bigcup_{0\leqslant s\leqslant 1} sC\Biggr) = \bigcup_{s>0} sC = \bigcup_{n\in\mathbb{N}^*} nC.$$

According to the Baire category theorem, the set $\bigcup_{n\in\mathbb{N}^*} nC$ has a non-empty norm-interior, if and only if, the norm-interior of one of the sets nC is non-empty. Equivalently,

$$\Biggl[\operatorname{Int}\bigl(\mathcal{B}(C^p)\bigr) = \operatorname{Int}\Biggl(\bigcup_{n\in\mathbb{N}^*} nC\Biggr) \neq \emptyset\Biggr] \quad \Leftrightarrow \quad [\operatorname{Int} C \neq \emptyset].$$

Making use of Proposition 2.1 in [1] and of Proposition 4 in [2], we deduce that, as C^p does not contain lines, the norm-interior of $\mathcal{B}(C^p)$ is non-empty if and only if there exists a sequence $g_n \in C^p$, such that

$$\lim_{n\to\infty}\|g_n\| = +\infty \quad \text{and} \quad \frac{g_n}{\|g_n\|} \rightharpoonup 0.$$

Then take $f_n = g_n/\|g_n\|$ to derive relation (1).

Conversely, if (a) fails, there are some point $x_0 \in C$ and some $\varepsilon > 0$ such that $x_0 + \varepsilon\mathbb{B}_X \subset C$. Let assume that there exists some sequence $(f_n)_{n\in\mathbb{N}^*}$ of norm one in X, weakly converging to 0 and satisfying $\sup_{x\in C}\langle f_n, x\rangle = 0$. Observe that

$$\begin{aligned}\sup_{x\in C}\langle f_n, x\rangle &\geqslant \sup_{b\in\mathbb{B}_X}\langle f_n, x_0 + \varepsilon b\rangle \\ &= \langle f_n, x_0\rangle + \varepsilon \sup_{b\in\mathbb{B}}\langle f_n, b\rangle \\ &\geqslant \langle f_n, x_0\rangle + \varepsilon.\end{aligned}$$

Using the fact that $f_n \rightharpoonup 0$ and passing to the limit in the last inequality as n tends to $+\infty$, we obtain $\lim_{n\to+\infty}\sup_{x\in C}\langle f_n, x\rangle \geqslant \varepsilon > 0$, a contradiction. □

Proposition 1 provides a simple proof of the following standard topological property which will be extensively used in the forthcoming sections.

Lemma 1. *Let C be a bounded closed and convex set with empty norm-interior which spans the real Hilbert space X. Let $h \in X$, $\|h\| = 1$ and let H be the closed hyperplane of X given by $H = \{x \in X\colon \langle h, x\rangle = 0\}$. Then, the projection of C on H spans H, and its relative norm-interior with respect to H is empty.*

Proof. As obviously $C \subseteq P_H C + \mathbb{R}h$, we deduce that

$$H + \mathbb{R}h = X = \overline{\mathrm{sp}}(C) \subseteq \overline{\mathrm{sp}}(P_H C) + \mathbb{R}h \subseteq H + \mathbb{R}h,$$

and therefore that $H \subseteq \overline{\mathrm{sp}}(P_H C) \subseteq H$. Thus $P_H C$, the projection of C on H, spans H. It remains to prove that the relative norm-interior of $P_H C$ with respect to H is empty.

Applying Proposition 1 to C we deduce that there exists a sequence $(f_n)_{n\in\mathbb{N}^*} \subset X$ with $\|f_n\| = 1$ fulfilling relation (1). Set $h_n = P_H f_n$; as $\|h_n - f_n\| = |\langle f_n, h\rangle|$, and $f_n \rightharpoonup 0$, we deduce that

$$\lim_{n\to\infty}\|h_n - f_n\| = 0. \tag{2}$$

On the one hand, the previous relation implies that

$$\lim_{n\to\infty}\|h_n\| = \lim_{n\to\infty}\|f_n\| = 1, \tag{3}$$

while on the other, using the fact that C is bounded ($C \subset \rho\mathbb{B}_X$ for some $\rho > 0$) and remarking that

$$\sup_{x\in C}\langle h_n, x\rangle \leqslant \sup_{x\in C}\langle f_n, x\rangle + \rho\|h_n - f_n\|, \tag{4}$$

and that $\langle h_n, x\rangle = \langle h_n, P_H x\rangle$ for every $x \in X$, we deduce from relations (4), (2) and (1) that

$$\lim_{n\to\infty}\Big(\sup_{x\in P_H C}\langle h_n, x\rangle\Big) = \lim_{n\to\infty}\Big(\sup_{x\in C}\langle h_n, x\rangle\Big) = \lim_{n\to\infty}\Big(\sup_{x\in C}\langle f_n, x\rangle\Big) = 0. \tag{5}$$

Finally, taking into account relations (2) and (1), we obtain that $h_n \rightharpoonup 0$.

Accordingly, the sequence $g_n = \frac{h_n}{\|h_n\|}$ satisfies relation (1) and by virtue of Proposition 1, applied to the closed convex subset $P_H C$ of the Hilbert space H, it follows that the norm-interior of $P_H C$ is empty. $\square$

3. Finite dimensional sets

Let us first characterize a class of closed and convex sets C for which we are able to construct a closed and convex set D, disjoint from C but which cannot be separated from B by any closed hyperplane. This class consists of unbounded closed convex sets C of finite dimension, that span a finite dimensional linear manifold of an infinite dimensional Hilbert space X.

The following result shows that in an infinite dimensional real Hilbert space, the class of unbounded finite dimensional closed convex subsets do no longer enjoy the remarkable separation property which characterizes them in the finite dimensional setting.

Proposition 2. *Let X be an infinite dimensional real Hilbert space, and C be an unbounded closed and convex set such that $\overline{\mathrm{sp}}(C)$ is a finite dimensional linear manifold. Then, there exists a set D, a translate of a closed convex cone, which is disjoint from C, and satisfies the following relation*:

$$\sup_{x\in D}\langle f, x\rangle > \inf_{x\in C}\langle f, x\rangle, \quad \forall f \in X,\ f \neq 0. \tag{6}$$

Proof. Recall that an unbounded closed convex set which spans a finite dimensional linear manifold necessarily contains half-lines. Hence, pick $h \in C^{\infty}$, $h \neq 0$, and let $H = \{x \in X\colon\ \langle h, x\rangle = 0\}$ define the hyperplane orthogonal to h. Let also L denote the finite dimensional linear subspace of X parallel to the linear manifold $\overline{\mathrm{sp}}(C)$. Finally, set L_0 for the intersection between L and H (which, since $h \in L$, coincides with the projection of L on H).

As L_0 is a finite dimensional linear subspace of H, there exists a basis of H denoted by $B = \{b_i\colon\ i \in I\}$ and a finite dimensional subset I_0 of I such that $B_0 = \{b_i\colon\ i \in I_0\}$ is a basis of L_0. The set $B \setminus B_0$ is infinite, therefore it contains a countable subset, say $B_1 = \{c_k\colon\ k \in \mathbb{N}^*\}$.

Fix a point $\bar{x} \in C$ and if x_0 denotes its projection on H set

$$d = x_0 - \left(\sum_{k=1}^{\infty} \frac{c_k}{k}\right).$$

The set D defined by

$$D = d + \left\{x \in X\colon\ \langle h, x\rangle \geqslant 0, \text{and } \bigl|\langle c_k, x\rangle\bigr| \leqslant \frac{\langle h, x\rangle}{k^2}, \forall k \in \mathbb{N}^*\right\}$$

is by construction a translate of a closed convex cone and is suitable.

Let us show that the projections on H of C and D are disjoint. In order to estimate the projection of C on H, remark that $x_0 - \bar{x}$ is colinear to h, and thus $x_0 + L = \bar{x} + L$. Accordingly, C is included in the linear manifold $x_0 + L$ whose projection on H is $x_0 + L_0$.

Let us now consider the projection of D on H. For every point $x \in D - d$, and $k \in \mathbb{N}^*$ such that $k > 2\langle h, x\rangle$ we deduce from the definition of D that

$$\left|\langle c_k, x\rangle\right| \leqslant \frac{\langle h, x\rangle}{k^2} < \frac{1}{2k}.$$

Consequently, the projection on H of $D - d$ does not contain any element x of H fulfilling

$$\langle c_k, x\rangle = 1/k, \quad \forall k \in \mathbb{N}^*. \tag{7}$$

Remark, as B_0 and B_1 are disjoint, that all the elements of the linear manifold $(\sum_{k=1}^{k=\infty} \frac{c_k}{k}) + L_0$ satisfy relation (7). Hence the projection of $D - d$ on H does not intersect $x_0 - d + L_0$. Equivalently, $(x_0 + L_0) \cap P_H(D) = \emptyset$. We have thus proved that the projection on H of C is contained in the linear manifold $x_0 + L_0$, while the projection on H of D does not intersect this manifold. Hence, the closed and convex sets C and D are disjoint.

To establish relation (6), let us estimate, for all $f \in X$, one of the values $\sup_{x\in D}\langle f, x\rangle$, or $\inf_{x\in C}\langle f, x\rangle$. We begin with the case $\langle f, h\rangle > 0$. Since for each $t > 0$, $d + th \in D$, we have $\sup_{x\in D}\langle f, x\rangle = +\infty$. Take $f \in X$ such that $\langle f, h\rangle = 0$ and $b \in B$ satisfying $\langle f, b\rangle \neq 0$. If $b \in B \setminus B_1$, then $d + \mathbb{R}(h + b) \subset D$, while if $b \in B_1$, that is $b = c_k$ for some $k \in \mathbb{N}^*$, then $d + \mathbb{R}(h + c_k/k) \subset D$. In both cases, $\sup_{x\in D}\langle f, x\rangle = +\infty$. Finally, let $f \in X$, satisfying $\langle f, h\rangle < 0$, and remark that $\bar{x} + \mathbb{R}h \subset C$, whence $\inf_{x\in C}\langle f, x\rangle = -\infty$.

Consequently, for every element $f \in X$, either $\sup_{x\in D}\langle f, x\rangle = +\infty$, or $\inf_{x\in C}\langle f, x\rangle = -\infty$, and relation (6) follows. As a consequence, no closed hyperplane of X can separate C and D. $\square$

4. The case of a separable underlying space

This section is concerned with the study of the case when the unbounded closed and convex set C has an empty norm-interior and spans the separable real Hilbert space X.

In defining the closed convex set D which cannot be separated from C by means of a closed hyperplane of X, we distinguish two cases.

4.1. Case 1: C contains a half-line

The main result of this subsection reads as follows.

Proposition 3. *Let X be a separable real Hilbert space, which is spanned by a closed convex set C. Assume that the norm-interior of C is empty, and that its recession cone contains at least one half-line ($C^\infty \neq \{0\}$). Then, there exists a closed and convex set D, which is either a closed linear manifold, or a translate of a closed convex cone, such that D is disjoint from C, and satisfies*

$$\inf_{x\in C}\langle f, x\rangle < \sup_{x\in D}\langle f, x\rangle, \quad \forall f \in X,\ f \neq 0. \tag{8}$$

Proof. By virtue of Theorem 1 we may suppose (if necessary, after a translation of C) that $0 \in C$ and that relation

$$\inf_{x\in C}\langle f, x\rangle < 0 < \sup_{x\in C}\langle f, x\rangle \tag{9}$$

holds for every $f \in X$.

Pick h, an element of norm one from C^∞, and set $H = \{x \in X\colon \langle h, x\rangle = 0\}$ for the closed hyperplane orthogonal to h. The first step in constructing the closed and convex set D is to prove that the norm-interior with respect to H of $P_H C$, the projection of C on H, is empty.

Lemma 2. *Let C be a closed convex set such that $0 \in C$ and such that relation* (9) *holds. Then, for every $r > 0$, the set $C_r = C \cap r\mathbb{B}_X$ also satisfies relation* (9).

Proof of Lemma 2. Let $f \in X$; from relation (9) it follows that there are $\bar{a}_f, \bar{b}_f \in C$ such that

$$\langle f, \bar{a}_f\rangle < 0 < \langle f, \bar{b}_f\rangle. \tag{10}$$

Remark that from relation (10) it follows that $\|\bar{a}_f\| \neq 0$ and $\|\bar{b}_f\| \neq 0$. Then, set

$$\lambda_a := \min\left(1, \frac{r}{\|\bar{a}_f\|}\right) \quad \text{and} \quad \lambda_b := \min\left(1, \frac{r}{\|\bar{b}_f\|}\right).$$

By convexity of C, as $0, \bar{a}_f, \bar{b}_f \in C$ and $0 < \lambda_a, \lambda_b \leqslant 1$, we deduce that $\lambda_a\bar{a}_f, \lambda_b\bar{b}_f \in C$. Relation (10) yields

$$\langle f, \lambda_a\bar{a}_f\rangle = \lambda_a\langle f, \bar{a}_f\rangle < 0 < \lambda_b\langle f, \bar{b}_f\rangle = \langle f, \lambda_b\bar{b}_f\rangle,$$

and the desired conclusion results by remarking that $\|\lambda_a\bar{a}_f\|, \|\lambda_b\bar{b}_f\| \leqslant r$. □

As a consequence of Lemma 2 and of Theorem 1, we deduce that, for every $r > 0$, the bounded closed and convex set C_r spans H, and thus satisfies the hypothesis of Lemma 1. The norm-interior with respect to H of the closed and convex set $P_H(C_r)$ is therefore empty. Obviously, $P_H C = \bigcup_{n\in\mathbb{N}^*} P_H(C_n)$, thus the projection on H of C is the union of a countable family of closed set of empty interior with respect to H. The desired conclusion follows once more from the Baire category theorem, as H is a complete metric space.

However, the projection on H of C is not always closed, and, although $P_H C$ has an empty norm-interior with respect to H, the closure of the projection of C on H may have a non-empty norm-interior with respect to H. We shall accordingly distinguish two cases.

Case (i): *the norm-interior with respect to H of $\overline{P_H C}$ is non-empty.* Hence, as the norm-interior of $P_H C$ is empty, there exist a point $\tilde{x} \in H \setminus P_H C$ and $\varepsilon > 0$ such that

$$\tilde{x} + \varepsilon\mathbb{B}_H \subseteq \overline{P_H C}. \tag{11}$$

In this case, consider the line $D = \tilde{x} + \mathbb{R}h$. The projection on H of D is the singleton $\{\tilde{x}\}$, and $\tilde{x} \notin P_H(C)$. Hence, $D \cap C = \emptyset$. Moreover, as relations

$$\langle f, x\rangle = \langle f, P_H x\rangle \tag{12}$$

and

$$\langle f, x\rangle = \langle f, \tilde{x}\rangle, \quad \forall x \in D, \tag{13}$$

hold for every $f \in H$, from (11), (12) and (13) we deduce that

$$\inf_{x\in C}\langle f,x\rangle=\inf_{x\in P_H C}\langle f,x\rangle\leqslant\langle f,\tilde{x}\rangle-\varepsilon\|f\|$$
$$<\langle f,\tilde{x}\rangle=\sup_{x\in D}\langle f,x\rangle,\quad \forall f\in H,\ f\neq 0. \tag{14}$$

Accordingly, relation (14) together with the fact that $\sup_{x\in D}\langle f,x\rangle=+\infty$ when $f\notin H$ yield relation (8).

Case (ii): *the norm-interior with respect to H of $\overline{P_H C}$ is empty.* Using again the Baire category theorem, we deduce that H is not the countable union of the closed sets $n\overline{P_H C}$, as each one of them have an empty interior with respect to H:

$$\bigcup_{r>0} r\overline{P_H C}=\bigcup_{n\in\mathbb{N}^*} n\overline{P_H C}\subsetneq H.$$

Pick

$$d\in\Bigl(H\setminus\bigcup_{r>0} r\overline{P_H C}\Bigr),$$

and set

$$D=d+\bigl\{x\in X\colon\ \langle h,x\rangle\geqslant 0 \text{ and } -P_H x\in\langle h,x\rangle\overline{P_H C}\bigr\}.$$

As claimed, D is the translate of a closed convex cone of X. The set $\bigcup_{r>0} r\overline{P_H C}$ is a convex cone, and hence

$$d\notin\bigcup_{r>0} r\overline{P_H C}=\bigcup_{r>0} r\overline{P_H C}+\bigcup_{r>0} r\overline{P_H C},$$

that is

$$\Bigl(d-\bigcup_{r>0} r\overline{P_H C}\Bigr)\cap\bigcup_{r>0} r\overline{P_H C}=\emptyset. \tag{15}$$

But $d-\bigcup_{r>0} r\overline{P_H C}$ is the projection on H of D, while the projection of C on H lies within $\bigcup_{r>0} r\overline{P_H C}$. Relation (15) implies thus that the projections on H of C and D are disjoint, and therefore $C\cap D=\emptyset$.

To establish relation (8) we will, similarly to the proof of Proposition 2, estimate, for all $f\in X$, one of the values $\sup_{x\in D}\langle f,x\rangle$, or $\inf_{x\in C}\langle f,x\rangle$. We begin with the case $\langle f,h\rangle>0$, remarking that, as $d+\mathbb{R}h\subset D$, it follows that $\sup_{x\in D}\langle f,x\rangle=+\infty$. Next, we consider $f\in X$ such that $\langle f,h\rangle=0$; from relations (9) and (12) it follows that there exists a point $\tilde{a}\in P_H C$ such that

$$\langle f,\tilde{a}\rangle>0. \tag{16}$$

For every $r>0$, the element $d+rh+r\tilde{a}$ belongs to D, and relation (16) yields

$$\sup_{r>0}\langle f,d+rh+r\tilde{a}\rangle=\sup_{r>0}\bigl(\langle f,d\rangle+r\langle f,\tilde{a}\rangle\bigr)=+\infty.$$

Therefore, $\sup_{x\in D}\langle f,x\rangle=+\infty$. Finally, let $f\in X$, satisfying $\langle f,h\rangle<0$. As $\mathbb{R}h\subset C$, we deduce that $\inf_{x\in C}\langle f,x\rangle=-\infty$.

Consequently, for every element $f\in X$, either $\sup_{x\in D}\langle f,x\rangle=+\infty$, or $\inf_{x\in C}\langle f,x\rangle=-\infty$, establishing relation (8). As a result, the sets C and D cannot be separated by a closed hyperplane of X. $\square$

4.2. Case 2: C is linearly bounded

The construction of a closed convex set D which cannot be separated from C by a closed hyperplane follows from the following lemma.

Lemma 3. *Let C be an unbounded linearly bounded closed and convex subset of a separable real Hilbert space X such that $0 \in C$. Let h be a non-zero element of X, and let H denote the closed hyperplane $\{x \in X\colon \langle h, x\rangle = 0\}$. If relation* (9) *holds, then a similar relation holds for the intersection between C and H:*

$$\inf_{x\in C\cap H} \langle f, x\rangle < 0 < \sup_{x\in C\cap H} \langle f, x\rangle, \quad \forall f \in H. \tag{17}$$

Moreover, the intersection $C \cap H$ remains an unbounded linearly bounded set.

Proof. For the purpose of obtaining a contradiction, suppose that relation (17) fails for some $f \in H$, $f \neq 0$. Then, by taking, if necessary, $-f$ instead of f, we may suppose that $\sup_{x\in C\cap H} \langle f, x\rangle = 0$ which means $H_0 = \{x \in H\colon \langle f, x\rangle \leqslant 0\}$ contains $C \cap H$.

Remark that, if a and b are two elements of C such that

$$\langle h, a\rangle < 0 < \langle h, b\rangle,$$

then, $H_0 + \mathbb{R}a = H_0 + \mathbb{R}b$. Indeed, set

$$c = \frac{\langle h, b\rangle}{\langle h, b\rangle - \langle h, a\rangle} a - \frac{\langle h, a\rangle}{\langle h, b\rangle - \langle h, a\rangle} b.$$

Obviously, by convexity, $c \in C$, and as $\langle h, c\rangle = 0$, it follows that $c \in C \cap H$. Hence $c \in H_0$ as well as

$$\frac{\langle h, b\rangle - \langle h, a\rangle}{-\langle h, a\rangle} c.$$

As

$$b = \frac{\langle h, b\rangle - \langle h, a\rangle}{-\langle h, a\rangle} c - \frac{\langle h, b\rangle}{\langle h, a\rangle} a,$$

it follows that $b \in H_0 + \mathbb{R}a$, which means that $H_0 + \mathbb{R}b \subseteq H_0 + \mathbb{R}a$. Reversing the role of a and b leads to the conclusion :

$$H_0 + \mathbb{R}b = H_0 + \mathbb{R}a. \tag{18}$$

Now we apply relation (9) for $h \in X$ and C to deduce that there are $\bar{a}_h, \bar{b}_h \in C$ such that $\langle h, \bar{a}_h\rangle < 0 < \langle h, \bar{b}_h\rangle$. According to relation (18), set H_1 for $H_0 + \mathbb{R}\bar{a}_h = H_0 + \mathbb{R}\bar{b}_h$. Let $x \in C \setminus H$. Then either $\langle h, x\rangle < 0$, and then relation (18) applied for x and $\bar{b}_h$ implies that $x \in H_1$, or $\langle h, x\rangle > 0$, and applying relation (18) to $\bar{a}_h$ and x, we deduce again that $x \in H_1$. The closed half-space H_1 of X contains C. As H_1 is an half-space, select $h_1, h_1 \neq 0$, such that $H_1 = \{x \in X\colon \langle h_1, x\rangle \leqslant 0\}$. Accordingly, $\sup_{x\in C} \langle h_1, x\rangle \leqslant 0$, which means that relation (9) fails for h_1. We obtain a contradiction, establishing in this way relation (17).

As the intersection between C and H is obviously linearly bounded, it remains to prove that $C \cap H$ is an unbounded set. To the end of obtaining a contradiction, let us suppose that the set $C \cap H$ is bounded, that is $C \cap H \subseteq r\mathbb{B}_X$ for some $r > 0$. As the set C is unbounded while its subset $C \cap H$ is bounded, we may select an unbounded sequence $(x_n)_{n\in\mathbb{N}^*}$ of C such that $x_n \notin C \cap H$. Accordingly, taking, if necessary, a subsequence, we may suppose that either $\langle h, x_n\rangle > 0$ for every $n \in \mathbb{N}^*$, or $\langle h, x_n\rangle < 0$ for every $n \in \mathbb{N}^*$.

Let us consider the case when, for every $n \in \mathbb{N}^*$, $\langle h, x_n\rangle > 0$ (the other case is similar), and set

$$y_n = \frac{\langle h, x_n\rangle}{\langle h, x_n\rangle - \langle h, \bar{a}_h\rangle}\bar{a}_h - \frac{\langle h, \bar{a}_h\rangle}{\langle h, x_n\rangle - \langle h, \bar{a}_h\rangle}x_n;$$

here $\bar{a}_h$ is one of the elements of C which satisfies $\langle h, \bar{a}_h\rangle < 0$ (its existence is guaranteed by Theorem 1 applied to C and $h \in X$).

Noticing that y_n is a convex combination of $\bar{a}_h \in C$ and $x_n \in C$, and satisfies the relation $\langle h, y_n\rangle = 0$, it follows that $y_n \in C \cap H$ for every $n \in \mathbb{N}^*$. Hence, $\|y_n\| \leqslant r$, and thus $\|\bar{a}_h - y_n\| \leqslant r + \|\bar{a}_h\|$, that is

$$\left\langle h, \frac{x_n - \bar{a}_h}{\|x_n - \bar{a}_h\|}\right\rangle \geqslant \frac{-\langle h, \bar{a}_h\rangle}{r + \|\bar{a}_h\|}, \quad \forall n \in \mathbb{N}^*. \tag{19}$$

Since on the one hand $\bar{a}_h$ and x_n belong to C, and on the other hand $(x_n)_{n\in\mathbb{N}^*}$ is an unbounded sequence, any weak cluster points, say $\overline{w}$, of the bounded sequence $(\frac{\bar{a}_h - x_n}{\|\bar{a}_h - x_n\|})$ belongs to C^∞. Moreover, as a result of relation (19), we have

$$\langle h, \overline{w}\rangle \geqslant \frac{-\langle h, \bar{a}_h\rangle}{R + \|\bar{a}_h\|} > 0,$$

relation which contradicts the fact that the set C is linearly bounded. This contradiction completes the proof of Lemma 3. □

When C is an unbounded, linearly bounded, closed and convex set with empty norm-interior and spans a separable real Hilbert space, we can now define a closed and convex set disjoint from C which cannot be separated from C by a closed hyperplane of X.

Proposition 4. *Let C be an unbounded, linearly bounded, closed and convex set with an empty norm-interior whose span X is a separable real Hilbert space. Then there exists a closed linear manifold D of X, disjoint from C such that*

$$\inf_{x\in C}\langle f, x\rangle < \sup_{x\in D}\langle f, x\rangle, \quad \forall f \in X,\ f \neq 0. \tag{20}$$

Proof. As we have already noticed, we may, without any loss of generality, suppose that $0 \in C$ and that relation (9) holds. Accordingly, by virtue of Lemma 2, we deduce that the set $C_r = C \cap r\mathbb{B}_X$ spans X for every $r > 0$.

Consequently, the set C_r satisfies the conditions of Lemma 1, while the set C satisfies the conditions of Lemma 3. By repeatedly applying the above mentioned lemmata we prove that for every closed linear subspace W of X of finite codimension, the set $P_W(C_r)$ spans W and its norm-interior with respect to W is empty, and the set $C \cap W$ is unbounded.

We use this result to define by induction two sequences $(a_n)_{n\in\mathbb{N}^*} \subset C$ and $(f_n)_{n\in\mathbb{N}^*} \subset X$, such that:

$$\langle a_i, f_j\rangle = \langle f_i, f_j\rangle = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j, \end{cases} \quad \forall i, j \in \mathbb{N}^*, \tag{21}$$

and

$$\sup_{x\in C_i}\langle f_i, x\rangle \leqslant \frac{1}{2^{i+1}}, \quad \forall i\in\mathbb{N}^*. \tag{22}$$

To construct a_1 and f_1, apply Proposition 1 to C_1 to deduce the existence of an element $g_1\in X$, $\|g_1\|=1$, such that $\sup_{x\in C_1}\langle g_1, x\rangle \leqslant (1/8)$. As C is an unbounded, linearly bounded set, $X\setminus\mathcal{B}(X)$ is dense in X. We can therefore find $f_1\in X$, $\|f_1\|=1$, such that $f_1\in X\setminus\mathcal{B}(C)$ and $\|g_1-f_1\|\leqslant 1/8$. Pick $a_1\in C$ such that $\langle f_1, a_1\rangle = 1$ (such an element a_1 exists as $f_1\in X\setminus\mathcal{B}(C)$). Accordingly, relation (21) is satisfied; as

$$\langle f_1, x\rangle \leqslant \langle g_1, x\rangle + \|f_1-g_1\|\|x\| \leqslant \frac{1}{4}, \quad \forall x\in C_1,$$

we deduce that relation (22) is satisfied as well.

Suppose that the sequences $(a_i)_{i\in\mathbb{N}^*}$ and $(b_i)_{i\in\mathbb{N}^*}$ have been defined up to a certain value $k\in\mathbb{N}^*$. Thus, for each i $(1\leqslant i<k)$, there are elements $a_i\in C$ and $f_i\in X$ such that relations (21) and (22) are satisfied for every $1\leqslant i, j<k$. Set for every $n\in\mathbb{N}^*$

$$W_n = \bigl\{x\in X\colon \langle x, a_i\rangle = \langle x, f_i\rangle = 0, 1\leqslant i<n\bigr\}.$$

As already remarked, on the one hand the projection $P_{W_k}(C_k)$ of C_k on W_k, is a bounded, closed and convex set which spans W_k and its norm-interior with respect to W_k is empty, and on the other $C\cap W_k$ is an unbounded, linearly bounded, closed and convex set. Applying Proposition 1 to $P_{W_k}(C_k)$ we deduce that there exists a point $g_k\in W_k$, $\|g_k\|=1$, such that

$$\sup_{x\in P_{W_k}(C_k)}\langle g_k, x\rangle \leqslant \frac{1}{2^{k+2}}. \tag{23}$$

Moreover since, $C\cap W_k$ is an infinite, unbounded, linearly bounded, closed and convex set, the set $W_k\setminus\mathcal{B}(C\cap W_k)$ is dense in W_k; whence there exists $f_k\in W_k$ such that $\|f_k\|=1$ and $\|f_k-g_k\|\leqslant 1/(k2^{k+2})$. It follows that

$$\begin{aligned}\langle f_k, x\rangle = \langle f_k, P_{W_k}x\rangle &\leqslant \langle g_n, P_{W_k}x\rangle + \|f_k-g_k\|\|P_{W_k}x\| \\ &\leqslant \frac{1}{2^{k+2}} + \frac{1}{k2^{k+2}}k = \frac{1}{2^{k+1}}, \quad \forall x\in C_k,\end{aligned}$$

and thus f_k satisfies relation (22) for $i=k$.

Finally, as $f_n\in W_k\setminus(C\cap W_k)$, there exist a point $a_k\in C\cap W_k$ such that $\langle a_k, f_k\rangle = 1$, which means that relation (21) is satisfied when $i=j=k$. As $a_k, f_k\in W_k$, the definition of the linear manifold W_k implies that relation (21) is also satisfied when $1\leqslant i<k$ and $j=k$. Hence the existence of the sequences $(a_i)_{i\in\mathbb{N}^*}$ and $(f_i)_{i\in\mathbb{N}^*}$ is achieved.

Now, set

$$D = \left\{x\in X\colon \langle f_i, x\rangle = \frac{1}{2^i}\right\}.$$

Remark that if

$$\bar{x} = \sum_{i\in\mathbb{N}^*}\frac{1}{2^i}f_i,$$

then, $\bar{x} \in D$, and consequently, D is a non-empty, closed linear manifold of X.

Let $x \in D$. From the definition of D, it follows that $\langle f_i, x\rangle = 1/2^i$ for every $i \in \mathbb{N}^*$ and from relation (22), as $\langle f_i, y\rangle \leqslant 1/2^{k+1}$ for every $y \in C_i$, we know that $x \notin C_i$ for every $i \in \mathbb{N}^*$. Accordingly, C and D are two disjoint closed and convex subsets of X.

Let us notice that $l(D)^\perp = \overline{\mathrm{sp}}(\{f_i\colon i \in \mathbb{N}^*\})$. Hence, for every $f \in l(D)^\perp$, there exists a sequence $(\alpha_i)_{i\in\mathbb{N}^*} \subset \mathbb{R}$ such that

$$f = \sum_{i\in\mathbb{N}^*} \alpha_i f_i.$$

Hence, for every $x \in D$ and $f \in l(D)^\perp$ we have

$$\langle f, x\rangle = \langle f, \bar{x}\rangle = \left\langle \sum_{i\in\mathbb{N}^*} \alpha_i f_i, \sum_{i\in\mathbb{N}^*} \frac{1}{2^i} f_i \right\rangle = \sum_{i\in\mathbb{N}^*} \frac{\alpha_i}{2^i}. \tag{24}$$

Notice that $0 \in C$ and $\langle f, 0\rangle = 0$. Moreover, $a_n \in C$ for every $n \in \mathbb{N}$, and

$$\langle f, a_n\rangle = \left\langle \sum_{i\in\mathbb{N}^*} \alpha_i f_i, a_n \right\rangle = \alpha_n.$$

Thus,

$$\inf_{x\in C} \langle f, x\rangle \leqslant 0 \quad \text{and} \quad \inf_{x\in C} \langle f, x\rangle \leqslant \alpha_n, \quad \forall n \in \mathbb{N}^*.$$

If $a = \min(0, \inf_{i\in\mathbb{N}^*} \alpha_i)$, then we obtain:

$$\inf_{x\in C} \langle f, x\rangle \leqslant a. \tag{25}$$

Straightforward calculations yield:

$$a < \sum_{i\in\mathbb{N}^*} \frac{\alpha_i}{2^i}. \tag{26}$$

Combining relations (24), (25) and (26), we obtain relation (20) for every $f \in l(D)^\perp$, $f \neq 0$. Finally, remark that for every $f \notin l(D)^\perp$, we have

$$\inf_{x\in C} \langle f, x\rangle < +\infty = \sup_{x\in D} \langle f, x\rangle.$$

We have finally constructed a closed linear manifold D disjoint from C which cannot be separated from C by any closed hyperplane of X. Therefore, Proposition 4 is proved. □

5. The general case

On the basis of the propositions established in the last two sections, we may finally state and prove the main result of this paper.

Theorem 2. *Let X be a real Hilbert space, and C be an unbounded closed and convex subset of X which is not CEL, that is with either an infinite codimension or an empty* *relative norm-interior. Then, there exists an unbounded closed and convex set D which does not intersect C such that*

$$\inf_{x\in C}\langle f,x\rangle < \sup_{x\in D}\langle f,x\rangle, \quad \forall f\in X,\ f\neq 0. \tag{27}$$

Moreover, the set D may be chosen to be either a closed linear manifold, or the translate of a closed convex cone of X.

Proof. If $\overline{\mathrm{sp}}(C)$ has a finite dimension, then Theorem 2 follows directly from Proposition 2. When the dimension of C is infinite, in order to apply Propositions 3 or 4, we use the following result, which reduces the problem to a separable Hilbert space setting.

Lemma 4. *Let C be an unbounded closed and convex set such that $\overline{\mathrm{sp}}(C)$ is infinite-dimensional, with either an infinite codimension or an empty relative norm-interior. Then, there exists a closed linear separable subspace Y of X, such that the projection $P_Y C$ of C on Y is unbounded, has an empty norm-interior with respect to Y, and spans Y.*

Proof of Lemma 4. Without any loss of generality, we may assume that $0\in C$. For technical reasons, the proof of Lemma 4 will split into two cases: (a) C is an unbounded closed convex set with an empty relative norm-interior, and (b) C is an unbounded closed convex set of infinite codimension and with a non-empty relative norm-interior.

Case (a): *C is an unbounded closed and convex set of empty relative norm-interior.* As $0\in C$, $\overline{\mathrm{sp}}(C)$, which in general is only a closed linear manifold, is a closed linear subspace of X. Consider C as a closed convex subset of the real Hilbert space $\overline{\mathrm{sp}}(C)$: C spans $\overline{\mathrm{sp}}(C)$, and the norm-interior of C with respect to $\overline{\mathrm{sp}}(C)$ is empty. Accordingly, we may apply Proposition 1 to deduce the existence of a sequence $(f_n)_{n\in\mathbb{N}^*}\subset\overline{\mathrm{sp}}(C)$, $\|f_n\|=1$, such that

$$\lim_{n\to\infty}\sup_{x\in C}\langle f,x\rangle = 0 \quad \text{and} \quad f_n \rightharpoonup 0.$$

Since C is unbounded, we may pick $f_0\in\overline{\mathrm{sp}}(C)$ such that $\sup_{x\in C}\langle f_0,x\rangle=+\infty$, and define the desired closed linear separable subspace Y of X by $Y=\overline{\mathrm{sp}}(\{f_i\colon i\in\mathbb{N}\})$.

Let us prove that $P_Y C$ is unbounded, spans Y, and that its norm-interior with respect to Y is empty. To this end, remarking that

$$\sup_{x\in P_Y C}\langle f_0,x\rangle = \sup_{x\in C}\langle f_0,x\rangle = +\infty,$$

it results that $P_Y C$ is unbounded.

As C is a convex set, and as $0\in C$, it follows that $C_{nr}\subseteq nC_r$, and thus,

$$C=\bigcup_{n\in\mathbb{N}^*}C_{nr}\subseteq\bigcup_{n\in\mathbb{N}^*}nC_r,$$

for every $n\in\mathbb{N}^*$ and $r>0$. Moreover, as relation $\overline{\mathrm{sp}}(nK)=\overline{\mathrm{sp}}(K)$ holds for every set K which contains the origin, we have

$$\overline{\mathrm{sp}}(C)=\overline{\mathrm{sp}}\Bigg(\bigcup_{n\in\mathbb{N}^*}nC_r\Bigg)=\bigcup_{n\in\mathbb{N}^*}\overline{\mathrm{sp}}(nC_r)=\overline{\mathrm{sp}}(C_r)\subseteq\overline{\mathrm{sp}}(C).$$

Hence, $\overline{\mathrm{sp}}(C) = \overline{\mathrm{sp}}(C_r)$ for every $r > 0$. Using the previous equality we deduce that

$$Y + \big(Y^{\perp} \cap \overline{\mathrm{sp}}(C)\big) = \overline{\mathrm{sp}}(C) = \overline{\mathrm{sp}}(C_r) \subset \overline{\mathrm{sp}}\big(P_Y(C_r)\big) + \big(Y^{\perp} \cap \overline{\mathrm{sp}}(C)\big).$$

Therefore, $Y \subseteq \overline{\mathrm{sp}}(P_Y(C_r))$. Accordingly, the closed and convex set $P_Y(C_r)$ spans Y.

Finally, remark that $f_n \in Y$, $\|f_n\| = 1$, $f_n \rightharpoonup 0$ and

$$\lim_{n\to\infty} \sup_{x\in P_Y(C_r)} \langle f, x\rangle = \lim_{n\to\infty} \sup_{x\in C_r} \langle f, x\rangle \leqslant \lim_{n\to\infty} \sup_{x\in C} \langle f, x\rangle = 0.$$

Thus, if we apply Proposition 1 to the closed convex subset $P_Y(C_r)$, we establish that the norm-interior with respect to Y of $P_Y(C_r)$ is empty. The projection on Y of C is thus a countable union of closed sets of empty norm-interior:

$$P_Y C = P_Y\Bigg(\bigcup_{n\in\mathbb{N}^*} C_n\Bigg) = \bigcup_{n\in\mathbb{N}^*} P_Y(C_n).$$

By the Baire category theorem, we derive that the norm-interior of $P_Y C$ with respect to Y is empty.

Case (b): *C is an unbounded closed and convex set with a non-empty relative norm-interior and C has an infinite dimension and codimension.* As C is unbounded, there exists a point $g_0 \in \overline{\mathrm{sp}}(C)$, $\|g_0\| = 1$ such that $\sup_{x\in C}\langle g_0, x\rangle = +\infty$. As $\overline{\mathrm{sp}}(C)$ is infinite dimensional, we may pick a sequence $(v_k)_{k\in\mathbb{N}^*} \subset \overline{\mathrm{sp}}(C)$ such that

$$\langle v_i, v_j\rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

and $v_1 = g_0$. Similarly, as $\overline{\mathrm{sp}}(C)^{\perp}$ is also an infinite dimensional closed linear subspace of X, we select a sequence $(w_p)_{p\in\mathbb{N}^*} \subset \overline{\mathrm{sp}}(C)^{\perp}$ such that

$$\langle w_i, w_j\rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Let define

$$f_i := \cos\left(\frac{1}{i}\right) w_i + \sin\left(\frac{1}{i}\right) v_i, \quad \forall i \in \mathbb{N}^*,$$

and set $Y := \overline{\mathrm{sp}}(\{f_i\colon i \in \mathbb{N}^*\})$.

We have to prove that $P_Y C$ is unbounded, spans Y, and that its norm-interior with respect to Y is empty. Let us first remark that, as $\langle w_i, x\rangle = 0$ for every $i \in \mathbb{N}^*$ and $x \in \overline{\mathrm{sp}}(C)$, we have

$$\sup_{x\in P_Y C} \langle f_1, x\rangle = \sup_{x\in C}\langle f_1, x\rangle = \sin(1) \sup_{x\in C}\langle v_1, x\rangle = +\infty,$$

and thus $P_Y C$ is unbounded.

In order to prove that $P_Y C$ spans Y, pick $\bar{x}$ in the relative norm-interior of C. Accordingly, there exists a real $\varepsilon > 0$ such that $\bar{x} + \varepsilon v_i \in C$ for every $i \in \mathbb{N}^*$. Let us notice that $(f_i)_{i\in\mathbb{N}^*}$ is basis for the Hilbert space Y, and that for every $f \in Y$, there exists a sequence $(\beta_i)_{i\in\mathbb{N}^*} \subset \mathbb{R}$ such that

$$f = \sum_{i\in\mathbb{N}^*} \beta_i f_i.$$

For every $f \neq 0$, there exists at least one value $\bar{k} \in \mathbb{N}^*$ such that $\beta_{\bar{k}} \neq 0$. Then

$$\langle f, \bar{x} + \varepsilon v_k \rangle - \langle f, \bar{x} \rangle = \varepsilon \beta_{\bar{k}} \neq 0;$$

hence the set $\langle f, P_Y C \rangle$ does not reduce to a singleton for any $f \in Y$, $f \neq 0$. By virtue of Theorem 1, $P_Y C$ spans Y.

Finally, observe that

$$P_Y\big(\overline{\text{sp}}(C)\big) = \left\{ f = \sum_{i \in \mathbb{N}^*} \beta_i f_i \colon \sum_{i \in \mathbb{N}^*} \left(\frac{\beta_i}{\sin(1/i)} \right)^2 < +\infty \right\},$$

is a linear subspace of Y which is both proper and dense, and thus of empty norm-interior with respect to Y. As $P_Y(C)$ is a part of $P_Y(\overline{\text{sp}}(C))$, its norm-interior with respect to Y must also be empty. $\square$

Once the existence of a separable closed subspace Y is achieved, let us return to proof of Theorem 2. In order to construct the closed convex set D, we will distinguish two cases, depending whether the norm-interior with respect of Y of $\overline{P_Y C}$ (the norm-closure of $P_Y C$), is empty or not.

Case (i): *the norm-interior of $\overline{P_Y C}$ with respect to Y is not empty.* From Lemma 4, it follows that the norm-interior of $P_Y C$ with respect to Y is empty. Accordingly, there exist a point $\tilde{x} \in Y \setminus P_Y C$ and a real $\varepsilon > 0$ such that

$$\tilde{x} + \varepsilon \mathbb{B}_Y \subset \overline{P_Y C}.$$

In this case, the closed linear manifold $D = \tilde{x} + Y^{\perp}$ will do the job. Indeed, as $\tilde{x} \notin P_Y C$, the projection on Y of D, that is the singleton $\{\tilde{x}\}$, is disjoint from $P_Y C$. Consequently, $D \cap C = \emptyset$.

Moreover, $l(D)^{\perp} = Y$, and thus,

$$\inf_{x \in C} \langle f, x \rangle < +\infty = \sup_{x \in D} \langle f, x \rangle, \quad \forall f \in X \setminus l(D)^{\perp} = X \setminus Y. \tag{28}$$

On the other hand, we have

$$\langle f, x \rangle = \langle f, \tilde{x} \rangle, \quad \forall x \in D,\ f \in Y, \tag{29}$$

and

$$\inf_{x \in C} \langle f, x \rangle = \inf_{x \in \overline{P_Y C}} \langle f, x \rangle \leqslant \langle f, \tilde{x} \rangle - \varepsilon \| f \| < \langle f, \tilde{x} \rangle, \quad \forall f \in Y,\ f \neq 0. \tag{30}$$

Therefore, relations (29) and (30) infer

$$\inf_{x \in C} \langle f, x \rangle \leqslant \langle f, \tilde{x} \rangle - \varepsilon \| f \| < \sup_{x \in D} \langle f, x \rangle, \quad \forall f \in Y, f \neq 0. \tag{31}$$

Relation (27) follows from relations (28) and (31).

Case (ii): *the norm-interior of $\overline{P_Y C}$ with respect to Y is empty.* Accordingly, $\overline{P_Y C}$ is a closed convex subset with an empty norm-interior in the separable real Hilbert space Y. As $\overline{P_Y C}$ contains $P_Y C$ which, by virtue of Lemma 4, spans Y and is unbounded, it follows that $\overline{P_Y C}$ is unbounded and spans Y.

We can now apply Proposition 3 or Proposition 4, depending whether $\overline{P_Y C}$ admits a recession half-line or not, and deduce that there exists a closed linear manifold $\tilde{D}$ of Y or a translate of a closed convex cone from Y, such that D is disjoint from C and

$$\inf_{x \in C} \langle f, x\rangle < \sup_{x \in \tilde{D}} \langle f, x\rangle \quad \forall f \in Y,\ f \neq 0. \tag{32}$$

In this case, let consider the closed and convex set $D = \tilde{D} + Y^{\perp}$. The projection on Y of D is $\tilde{D}$ which, by virtue of Propositions 3 or 4, is disjoint from $P_Y C$. Hence $D \cap C = \emptyset$. Finally, relation (27) follows from relation (32) when $f \in Y$, $f \neq 0$, and from the relation

$$\inf_{x \in C} \langle f, x\rangle < +\infty = \sup_{x \in D} \langle f, x\rangle$$

valid for every $f \notin Y$. $\square$

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